Set families with a forbidden pattern

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Abstract

A *balanced pattern* of order $2d$ is an element $P \in \{+,-\}^{2d}$, where both signs appear d times. Two sets $A, B \subset [n]$ form a P-pattern, which we denote by $\mathrm{pat}(A,B) = P$, if $A \triangle B = \{j_1, \ldots, j_{2d}\}$ with $1 \leq j_1 < \cdots < j_{2d} \leq n$ and $\{i \in [2d] : P_i = +\} = \{i \in [2d] : j_i \in A \setminus B\}$. We say $\mathcal{A} \subset \mathcal{P}[n]$ is P-free if $\mathrm{pat}(A,B) \neq P$ for all $A, B \in \mathcal{A}$. We consider the following extremal question: how large can a family $\mathcal{A} \subset \mathcal{P}[n]$ be if $\mathcal{A}$ is P-free?

We prove a number of results on the sizes of such families. In particular, we show that for some fixed $c > 0$, if P is a d-balanced pattern with $d < c \log\log n$ then $|\mathcal{A}| = o(2^n)$. We then give stronger bounds in the cases when (i) P consists of d + signs, followed by d − signs and (ii) P consists of alternating signs. In both cases, if $d = o(\sqrt{n})$ then $|\mathcal{A}| = o(2^n)$. In the case of (i), this is tight.

1 Introduction

A central goal in extremal set theory is to understand how large a set family can be subject to some restriction on the intersections of its elements. Given $\mathcal{L} \subset \mathbb{N} \cup \{0\}$, we say that a set family $\mathcal{A}$ is $\mathcal{L}$-intersecting if $|A \cap B| \in \mathcal{L}$ for all distinct $A, B \in \mathcal{A}$. Taking $\mathcal{L}_t = \{s \in \mathbb{N} : s \geq t\}$, a fundamental theorem of Erdős, Ko and Rado [6] shows that $\mathcal{L}_t$-intersecting families $\mathcal{A} \subset \binom{[n]}{k}$ satisfy $|\mathcal{A}| \leq \binom{n-t}{k-t}$, provided $n \geq n_0(k,t)$. Another important theorem due to Frankl and Füredi [8] shows that if $\mathcal{L}_{\ell,\ell'} := \{s < \ell \text{ or } s \geq k - \ell'\}$, then any $\mathcal{L}_{\ell,\ell'}$-intersecting family $\mathcal{A} \subset \binom{[n]}{k}$ satisfies $|\mathcal{A}| \leq cn^{\max(\ell,\ell')}$, for some constant c depending on k, ℓ and ℓ'. See [2], [3], [7], [9] for an overview of this extensive topic.

Here we are concerned with understanding the effect of restricting the *pattern* formed between elements of a set family. A *difference pattern* or *pattern* of order t is an element $P \in \{+,-\}^t$. Given such a pattern P, let $S_+(P) = \{i \in [t] : P_i = +\} \subset [t]$ and $s_+(P) = |S_+(P)|$. Define $S_-(P)$ and $s_-(P)$ analogously. Two sets $A, B \subset [n]$ form a *difference pattern* P if:

(i) $A \triangle B = \{j_1, \ldots, j_t\}$ with $j_1 < \cdots < j_t$, and

(ii) $\{i \in [t] : P_i = +\} = \{i \in [t] : j_i \in A \setminus B\}$.

We denote this by writing $\mathrm{pat}(A,B) = P$. A family of subsets $\mathcal{A} \subset \mathcal{P}[n]$ is *P-free* if $\mathrm{pat}(A,B) \neq P$ for all distinct $A, B \in \mathcal{A}$. In this paper we consider the following natural question: given a pattern P, how large can a family $\mathcal{A} \subset \mathcal{P}[n]$ be if it is P-free?

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First note the following simple observation. If $s_+(P) \neq s_-(P)$ then large P-free families exist. Indeed, if $|s_+(P) - s_-(P)| = m > 0$ then the following families are P-free:

$$\mathcal{B}_1 = \{A \subset [n] : |A| \in [0, m-1] \mod 2m\}; \quad \mathcal{B}_2 = \{A \subset [n] : |A| \in [m, 2m-1] \mod 2m\}.$$

Clearly either $|\mathcal{B}_1| \geq 2^{n-1}$ or $|\mathcal{B}_2| \geq 2^{n-1}$. We will therefore focus on the case when $s_+(P) = s_-(P) = d$. We say that such patterns are *d-balanced*. For a balanced pattern P it is only possible that $\text{pat}(A, B) = P$ if $|A| = |B|$. Thus, our question on balanced patterns essentially reduces to a question for uniform families. Given $0 \leq k \leq n$, define

$$f(n, k, P) := \max\left\{|\mathcal{A}| : P\text{-free families } \mathcal{A} \subset \binom{[n]}{k}\right\}.$$

Let $f(n, k, d) = \max\{f(n, k, P) : P \text{ is } d\text{-balanced}\}$. We will also write $\delta(n, k, P)$ and $\delta(n, k, d)$ for the corresponding extremal densities, i.e. $\delta(n, k, P) := f(n, k, P)/\binom{n}{k}$, and $\delta(n, k, d) := f(n, k, d)/\binom{n}{k}$. Note also that if $\mathcal{A} \subset \binom{[n]}{k}$ is P-free then the family $\mathcal{A}^c = \{[n] \setminus A : A \in \mathcal{A}\} \subset \binom{[n]}{n-k}$ is also P-free. Therefore $f(n, k, P) = f(n, n-k, P)$ and it suffices to bound $f(n, k, P)$ for $k \leq n/2$.

Our first aim is to prove a density result for d-balanced patterns of small order. That is, we will show that for fixed d, any sequence of integers $\{k_n\}_{n=1}^{\infty}$ tending to infinity with n with $k_n \leq n/2$ satisfies $\lim_{n\to\infty} \delta(n, k_n, d) = 0$. The condition that k is not fixed and tends to infinity with n will be crucial. This is different from the case in the Frankl-Füredi Theorem, which tells us that we can take some fixed $k \geq 2d-1$, $\ell = k-d$ and $\ell' = d-1$, and if $\mathcal{A} \subset \binom{[n]}{k}$ with $|\mathcal{A}| = \omega(n^{k-d})$ then there are $A, B \in \mathcal{A}$ with $|A \triangle B| = 2d$, i.e. A and B form a P-pattern for *some* d-balanced pattern P. Indeed, take any fixed $k := k(d)$, and consider the family $\mathcal{A}_0 \subset \binom{[n]}{k}$ given by

$$\mathcal{A}_0 = \left\{A \subset [n] : \left|A \cap \left(\frac{(i-1)n}{k}, \frac{in}{k}\right]\right| = 1 \text{ for all } i \in [k]\right\}.$$

Then $|\mathcal{A}_0| \geq c_k n^k$ for some absolute constant $c_k > 0$, but it is easily seen that $\mathcal{A}_0$ does not contain the pattern $++--$. Therefore, there does not exist a density theorem for d-balanced patterns in subsets of $\binom{[n]}{k}$ with fixed k, as in the Frankl-Füredi theorem.

Our first result shows that such a density theorem does hold for k growing with n.

Theorem 1. *Given $d, k, n \in \mathbb{N}$ with $2k \leq n$ and taking $a_d = (8d)^{5d}$ and $c_d = 6d8^{-d}$ we have*

$$\delta(n, k, d) \leq a_d k^{-c_d}.$$

By our discussion above for fixed k we see that Theorem 1 is in a sense a 'high-dimensional' result. Also note that Theorem 1 shows there is a constant $c > 0$ with the property that if P is a d-balanced pattern with $d \leq c \log\log n$ and $\mathcal{A} \subset \mathcal{P}[n]$ which is P-free, then $|\mathcal{A}| = o(2^n)$.

Let $\text{IP}(d)$ denote the d-balanced pattern consisting of d plus signs, followed by d minus signs. We refer to these as *interval patterns*. Given the obstruction of $\text{IP}(2)$ above, it is natural to ask for bounds on $f(n, k, \text{IP}(d))$.

Theorem 2. *Given $d, k, n \in \mathbb{N}$ with $2k \leq n$ we have*

$$\delta(n, k, \text{IP}(d)) = O(d^2 k^{-1}).$$

In particular, families $\mathcal{A} \subset \mathcal{P}[n]$ which are IP(d)-free for all $d = o(\sqrt{n})$ satisfy $|\mathcal{A}| = o(2^n)$. Furthermore, this turns out to be tight – if $d \geq c\sqrt{n}$ then there are IP(d)-free families $\mathcal{A} \subset \mathcal{P}[n]$ with $|\mathcal{A}| = \Omega_c(2^n)$.

Lastly, we consider the d-balanced pattern AP(d) consisting of alternating plus and minus signs, e.g. AP$(2) = + - + -$. We refer to these as *alternating patterns*. Our next result proves a density result for such patterns.

Theorem 3. *Given $d, k, n \in \mathbb{N}$ with $2k \leq n$ we have*

$$\delta(n, k, \mathrm{AP}(d)) = O\Big(\log^{-1}\Big(\frac{k}{d^2}\Big)\Big).$$

Thus again, all families $\mathcal{A} \subset \mathcal{P}[n]$ which are AP(d)-free for $d = o(\sqrt{n})$ satisfy $|\mathcal{A}| = o(2^n)$. Unlike in the case of the interval patterns, we do not know if this is tight.

Before closing the introduction, we mention some further results related to this topic. A family $\mathcal{A} \subset \mathcal{P}[n]$ is said to be a *tilted Sperner family* if for all distinct $A, B \in \mathcal{A}$ we have $|B \setminus A| \neq 2|A \setminus B|$. Equivalently, $\mathcal{A}$ is P-free for all patterns P with $|S_-(P)| = 2|S_+(P)|$. Kalai raised the question of how large a tilted Sperner family $\mathcal{A} \subset \mathcal{P}[n]$ can be. In [13], Leader and the second author proved that such families satisfy $|\mathcal{A}| \leq (1 + o(1))\binom{n}{n/2}$, which is asymptotically optimal. For sufficiently large n, the extremal families were also determined. In [14], the second author proved that this bound almost still applies if we only forbid 'tilted pairs' A, B with a single pattern. It was shown that if $\mathcal{A} \subset \mathcal{P}[n]$ does not contain $A, B \in \mathcal{P}[n]$ with $|B \setminus A| \neq 2|A \setminus B|$ for all distinct $A, B \in \mathcal{A}$ and satisfying $a < b$ for all $a \in A \setminus B$ and $b \in B \setminus A$ then $|\mathcal{A}| \leq C^{\sqrt{\log n}}\binom{n}{n/2}$, for some constant $C > 0$. This condition is equivalent to $\mathcal{A}$ being $P(d)$-free for all patterns $P(d)$ consisting of d $+$ signs followed by $2d$ $-$ signs. This bound was recently improved by Gerbner and Vizer in [11]. They proved that such families satisfy $|\mathcal{A}| \leq C\sqrt{\log n}\binom{n}{n/2}$. No family is known for this problem with order more than $C\binom{n}{n/2}$.

Lastly, we mention a fascinating question raised by Johnson and Talbot [12] related to Theorem 1 (similar conjectures have been raised by Bollobás, Leader and Malvenuto [4], and Bukh [5]). Our phrasing slightly differs from that in [12].

Question (Johnson–Talbot)**.** *Is it true that for any $k \in \mathbb{N}$ and $\alpha > 0$ there is $n_0(k, \alpha) \in \mathbb{N}$ with the following property. Suppose that $n \geq n_0(k, \alpha)$ and that $\mathcal{A} \subset \binom{[n]}{n/2}$ with $|\mathcal{A}| \geq \alpha\binom{n}{n/2}$. Then there are disjoint sets $S \in \binom{[n]}{n/2 - \lfloor k/2 \rfloor}$ and $T \in \binom{[n] \setminus S}{k}$ such that the family $\mathcal{C}_{T,S} := \big\{S \cup U : U \in \binom{T}{\lfloor k/2 \rfloor}\big\}$ is contained in $\mathcal{A}$.*

This is true for $k = 3$, but is already open for $k = 4$. In this case it is possible to guarantee that $|\mathcal{C}_{T,S} \cap \mathcal{A}| \geq 5$ for some T, S (note $|\mathcal{C}_{T,S}| = 6$ for $k = 4$). More generally, Johnson and Talbot [12] proved that under the hypothesis above, $|\mathcal{C}_{T,S} \cap \mathcal{A}| \geq 4 \cdot 3^{(k-4)/3}$ for some T, S. We note the conclusion that dense subsets of $\mathcal{P}[n]$ contain all small patterns (from Theorem 1) would immediately follow from a positive answer to this question. Indeed, for $k = 2d$ any set $\mathcal{C}_{T,S}$ contains every d-balanced pattern. Theorem 1 may be seen as giving (weak) evidence for the question: for $k = 2d$ and any d-balanced pattern P, there is T and S and sets $A, B \in \mathcal{C}_{T,S} \cap \mathcal{A}$ with $\mathrm{pat}(A, B) = P$.

Notation: Given a set X, we write $\mathcal{P}(X)$ for the power set of X and $\binom{X}{k} = \{A \subset X : |A| = k\}$. Given integers $m, n \in \mathbb{N}$ with $m \leq n$, we let $[n] = \{1, \ldots, n\}$ and $[m, n] = \{m, m+1, \ldots, n\}$. We also write $(n)_m$ for the falling factorial $(n)_m = n(n-1)\cdots(n-m+1)$.

2 Small balanced patterns

In this section we prove Theorem 1. We will find it convenient to prove many of our results restricted of the middle layer. We then simply write $f(k,P)$ for $f(2k,k,P)$, $\delta(k,P)$ for $\delta(2k,k,P)$, etc.. The following simple observation is useful to move results between different layers of the cube.

Proposition 2.1. *Let $n,m,k,l \in \mathbb{N}$ with $m \le n$, $l \le k$ and $k+m-l \le n$. Let P be a pattern. Then $\delta(n,k,P) \le \delta(m,l,P)$.*

Proof. Suppose $\mathcal{A} \subset \binom{[n]}{k}$ is P-free with $|\mathcal{A}| = \delta(n,k,P)\binom{n}{k}$. Select two disjoint sets T and U of order m and $k-l$ uniformly at random (possible as $k+m-l \le n$). Then let $\mathcal{A}_{T,U} = \{A \in \binom{T}{l} : A \cup U \in \mathcal{A}\}$. As $\mathcal{A}$ is P-free, the set $\mathcal{A}_{T,U}$ must also be P-free for all T,U, giving $|\mathcal{A}_{T,U}| \le \delta(m,l,P)\binom{m}{l}$. However, $\mathbb{E}_{T,U}|\mathcal{A}_{T,U}| = \delta(n,k,P)\binom{m}{l}$. The result follows. $\square$

Our next two lemmas are the main steps in the proof of Theorem 1. Combined they will allow a recursive bound for $\delta(k,d)$ based on bounds on $\delta(k',d')$ for $k' < k$ and $d' < d$.

Lemma 2.2. *Let $d,k \in \mathbb{N}$ with $k^{1/2} \ge 16\log k$ and let P be a d-balanced pattern with $P_1 \ne P_{2d}$. Then given any $\gamma \in [\frac{16\log k}{k^{1/2}}, 1]$ we have*

$$\delta(k,P) \le \max\left(\gamma, 6\sqrt{\delta\big(\lceil \gamma^2 k/64\rceil, d-1\big)}\right).$$

Proof. Let γ be chosen as above and let $\mathcal{A} \subset \binom{[2k]}{k}$ be P-free with $|\mathcal{A}| = \alpha\binom{2k}{k}$. If $\alpha \le \gamma$ then there is nothing to prove, so we will assume that $\alpha > \gamma \ge \frac{16\log k}{k^{1/2}}$. We will first show that there are many pairs $A,B \in \mathcal{A}$ with $|A \triangle B| = 2$. Indeed, given $C \in \binom{[2k]}{k+1}$ let y_C denote the number of $A \in \mathcal{A}$ with $A \subset C$. Then we have

$$\sum_{C \in \binom{[2k]}{k+1}} y_C = \left|\left\{(A,C) \in \mathcal{A} \times \binom{[2k]}{k+1} : A \subset C\right\}\right| = |\mathcal{A}|k \ge \alpha k \binom{2k}{k+1}.$$

As for every pair $A,B \in \mathcal{A}$ with $|A \triangle B| = 2$ there is a unique set $C \in \binom{[2k]}{k+1}$ with $A,B \subset C$, we obtain

$$\begin{aligned} \left|\left\{(A,B) \in \binom{\mathcal{A}}{2} : |A \triangle B| = 2\right\}\right| &= \sum_{C \in \binom{[2k]}{k+1}} \binom{y_C}{2} \ge \binom{2k}{k+1}\binom{\alpha k}{2} \\ &\ge \frac{\alpha^2 k^2}{4} \times \frac{2k(2k-1)}{(k+1)k}\binom{2k-2}{k-1} \ge \frac{\alpha^2 k^2}{2}\binom{2k-2}{k-1}. \end{aligned} \tag{1}$$

The first inequality holds by the convexity of $\binom{x}{2}$ and the second since $\alpha k - 1 \ge \alpha k/2$ as $\alpha \ge 2/k$. Now given $1 \le i < j \le 2k$, let $\mathcal{A}_{i,j} := \{A \in [2k] \setminus \{i,j\} : A \cup \{i\}, A \cup \{j\} \in \mathcal{A}\}$. Note that from (1) we have

$$\sum_{i<j} |\mathcal{A}_{i,j}| = \left|\left\{(A,B) \in \mathcal{A} \times \mathcal{A} : |A \triangle B| = 2\right\}\right| \ge \frac{\alpha^2 k^2}{2}\binom{2k-2}{k-1}. \tag{2}$$

Also let $\alpha_{i,j}$ and $\beta_{i,j}$ be defined so that $|\mathcal{A}_{i,j}| = \alpha_{i,j}\binom{2k-2}{k-1}$ and $\beta_{i,j} = (j-i)/2k$. By (2) we find $\{i,j\}$ with $\alpha_{i,j} \geq \frac{\alpha^2}{8}$ and $\beta_{i,j} \geq \frac{\alpha^2}{16}$. Indeed, we have

$$\sum_{\{i,j\}:\alpha_{i,j}<\frac{\alpha^2}{8}} |\mathcal{A}_{i,j}| + \sum_{\{i,j\}:\beta_{i,j}<\frac{\alpha^2}{16}} |\mathcal{A}_{i,j}| < \binom{2k}{2}\frac{\alpha^2}{8}\binom{2k-2}{k-1} + 2k \times \frac{\alpha^2}{16}2k\binom{2k-2}{k-1} \leq \frac{\alpha^2 k^2}{2}\binom{2k-2}{k-1}.$$

Combined with (2) we see that a claimed pair $\{i,j\}$ exists. Fix such a pair $\{i,j\}$ and set $\mathcal{B} = \mathcal{A}_{i,j}$. Now let $X = [i+1, j-1]$ and $Y = [n] \setminus [i,j]$ so that $\mathcal{B} \subset \binom{X\cup Y}{k-1}$. Partition elements from $\binom{X\cup Y}{k-1}$ according to how they intersect X, for each $\ell \in [0, j-i-2]$ letting

$$X_\ell = \Big\{A \in \binom{X \cup Y}{k-1} : \big|A \cap X\big| = \ell\Big\}.$$

Also let $\mathcal{B}_\ell = \mathcal{B} \cap X_\ell$ and $L = \Big\{\ell : \big|\ell - \frac{|X|}{2}\big| \leq \sqrt{|X|\log\left(\frac{8}{\alpha}\right)}\Big\}$. By Chernoff's inequality we have

$$\sum_{\ell \notin L} |X_\ell| \leq \frac{\alpha^2}{32}\binom{|X|+|Y|}{k-1}.$$

Using that $|\mathcal{B}| = \alpha_{i,j}\binom{|X|+|Y|}{k-1} \geq \frac{\alpha^2}{16}\binom{|X|+|Y|}{k-1}$ this shows that

$$\sum_{\ell \in L} |\mathcal{B}_\ell| \geq |\mathcal{B}| - \frac{\alpha^2}{32}\binom{|X|+|Y|}{k-1} \geq \frac{\alpha^2}{32}\binom{|X|+|Y|}{k-1} \geq \frac{\alpha^2}{32}\sum_{\ell \in L}|X_\ell|.$$

The last inequality here holds since the sets X_ℓ are disjoint subsets of $\binom{X\cup Y}{k-1}$. Thus for some $\ell \in L$ we have $|\mathcal{B}_\ell| \geq \frac{\alpha^2}{32}|X_\ell|$. By averaging, we find a set $U \subset Y$ with $|U| = k - \ell - 1$ such that the family $\mathcal{C} = \big\{C \in \binom{X}{\ell} : C \cup U \in \mathcal{B}_\ell\big\}$ satisfies $|\mathcal{C}| \geq \frac{\alpha^2}{32}\binom{|X|}{\ell}$.

To complete the proof, let Q denote the pattern obtained from P by removing P_1 and P_{2d}, i.e. $Q = P_2 \cdots P_{2d-1}$. Note that as $P_1 \neq P_{2d}$ we see that Q is $(d-1)$-balanced. We claim that $\mathcal{C}$ is Q-free. Indeed, suppose $C_1, C_2 \in \mathcal{C}$ with $\mathrm{pat}(C_1, C_2) = Q$. Then by definition of $\mathcal{C}$ and $\mathcal{B} = \mathcal{A}_{i,j}$ we have

$$\Big\{C_a \cup U \cup \{h\} : a \in \{1,2\}, h \in \{i,j\}\Big\} \subset \mathcal{A}.$$

If $P_1 = +$ we find $\mathrm{pat}(C_1 \cup U \cup \{i\}, C_2 \cup U \cup \{j\}) = P$. If $P_1 = -$ we find $\mathrm{pat}(C_1 \cup U \cup \{j\}, C_2 \cup U \cup \{i\}) = P$. Thus $\mathcal{C}$ must be Q-free and

$$\frac{\alpha^2}{32}\binom{|X|}{\ell} \leq |\mathcal{C}| \leq \delta(|X|, \ell, Q)\binom{|X|}{\ell}.$$

Take $k' = \Big\lfloor \frac{|X|}{2} - \sqrt{|X|\log\left(\frac{8}{\alpha}\right)}\Big\rfloor$. A calculation shows that $\frac{|X|}{4} \geq \sqrt{|X|\log\left(\frac{8}{\alpha}\right)} + 2$ since $\alpha \geq \frac{16\log k}{k^{1/2}}$ and $|X| + 2 = \beta_{i,j}2k \geq \frac{\alpha^2 k}{8}$. This gives

$$k' \geq \frac{|X|}{2} - \sqrt{|X|\log\left(\frac{8}{\alpha}\right)} - 1 \geq \frac{|X|}{4} + 1 \geq \left\lceil \frac{\beta_{i,j}k}{2} \right\rceil \geq \left\lceil \frac{\alpha^2 k}{64} \right\rceil \geq \left\lceil \frac{\gamma^2 k}{64} \right\rceil.$$

Since $\ell \in L$ we have $k' \leq \ell \leq |X| - k'$. Using Proposition 2.1 we find that $\frac{\alpha^2}{32} \leq \delta(|X|, \ell, d-1) \leq \delta(2k', k', d-1) = \delta(k', d-1) \leq \delta(\lceil\frac{\gamma^2 k}{64}\rceil, d-1)$. Rearranging this gives $\alpha \leq 6\sqrt{\delta(\lceil\frac{\gamma^2 k}{64}\rceil, d-1)}$. $\square$

Our second lemma deals with the case where P starts and ends with the same signs.

Lemma 2.3. *Let $d \in \mathbb{N}$ and let P be a d-balanced pattern with $P_1 = P_{2d}$. Then there are $d_1, d_2 \geq 1$ with $d_1 + d_2 = d$ such that the following holds. For every k_1, k_2 with $2k_1 + k_2 = k$ we have*

$$\delta(k, P) \leq \max\Big(2e^{-k_1/12}, 4\delta(k_1, d_1), 4(3k_1)^{2d_1}\delta(k_2, d_2)\Big).$$

Similarly for every k_1, k_2 with $k_1 + 2k_2 = k$ we have

$$\delta(k, P) \leq \max\Big(2e^{-k_2/12}, 4\delta(k_2, d_2), 4(3k_2)^{2d_2}\delta(k_1, d_1)\Big).$$

Proof. To begin, for each $\ell \in [0, 2d]$ let

$$c_\ell = \big|\{j \in [\ell] : P_j = +\}\big| - \big|\{j \in [\ell] : P_j = -\}\big|.$$

As P is d-balanced and $P_1 = P_{2d}$, we have $c_{2d-1} = -c_1$. Combined with the fact that $c_0 = c_{2d} = 0$ and c_ℓ changes by exactly 1 as ℓ increases, we see that $c_{2d_1} = 0$ for some $1 \leq d_1 \leq d-1$. Setting $d_2 := d - d_1$ and $Q_1 = P_1 \cdots P_{2d_1}$, $Q_2 = P_{2d_1+1} \cdots P_{2d}$ it is easy to see that these patterns are d_1-balanced and d_2-balanced respectively.

Now suppose that $\mathcal{A} \subset \binom{[2k]}{k}$ with $|\mathcal{A}| = \alpha\binom{2k}{k}$ and that $\mathcal{A}$ is P-free. We will prove the first bound above as the second bound is proved identically. We will assume that $\alpha \geq 2e^{-k_1/12}$ as otherwise there is nothing to show. Partition $[2k]$ into two consecutive intervals $I_1 = [3k_1]$ and $I_2 = [3k_1 + 1, 2k]$. For each $\ell \in I_1$ let $Z_\ell := \binom{I_1}{\ell} \times \binom{I_2}{k-\ell}$. Let $L = \Big\{\ell \in I_1 : |\ell - 3k_1/2| \leq \sqrt{3k_1 \log\left(\frac{2}{\alpha}\right)}\Big\}$. Note that as $|\bigcup_{\ell \notin L} Z_\ell| \leq \frac{\alpha}{2}\binom{2k}{k}$ by Chernoff's inequality, we have $|\mathcal{A} \cap Z_\ell| \geq \frac{\alpha}{2}|Z_\ell|$ for some $\ell \in L$. Fix such a choice of ℓ and set $Z := Z_\ell$ and $\mathcal{B} = \mathcal{A} \cap Z_\ell$ so that $\mathcal{B} \subset Z$ with $|\mathcal{B}| \geq \frac{\alpha}{2}|Z|$.

We will now prove that α satisfies

$$\alpha \leq \max\Big(4\delta(|I_1|, \ell, Q_1), 4|I_1|^{2d_1}\delta(|I_2|, k-\ell, Q_2)\Big). \tag{3}$$

To see this, we may assume that $\alpha \geq 4\delta(|I_1|, \ell, Q_1)$ as otherwise there is nothing to show. Consider the set $\mathcal{P}_{Q_1}$ given by

$$\mathcal{P}_{Q_1} = \Big\{(A, B) \in Z \times Z : \mathrm{pat}(A \cap I_1, B \cap I_1) = Q_1 \text{ and } A \cap I_2 = B \cap I_2\Big\}.$$

We will first show that $|(\mathcal{B} \times \mathcal{B}) \cap \mathcal{P}_{Q_1}| \geq \frac{\alpha}{4|I_1|^{2d_1}}|\mathcal{P}_{Q_1}|$. Indeed, for each $D \in \binom{I_2}{k-\ell}$ let

$$\mathcal{E}(D) := \Big\{C \in \binom{I_1}{\ell} : C \cup D \in \mathcal{B}\Big\}; \qquad \mathcal{P}_{Q_1}(D) := \big\{C, C' \in \mathcal{E}(D) : \mathrm{pat}(C, C') = Q_1\big\}.$$

Noting that each $\mathcal{C} \subset \binom{I_1}{\ell}$ with $|\mathcal{C}| > \delta(|I_1|, \ell, Q_1)\binom{|I_1|}{\ell}$ contains C, C' with $\mathrm{pat}(C, C') = Q_1$, we find $|\mathcal{P}_{Q_1}(D)| \geq |\mathcal{E}(D)| - \delta(|I_1|, \ell, Q_1)\binom{|I_1|}{\ell}$. Combined these give

$$\big|(\mathcal{B} \times \mathcal{B}) \cap \mathcal{P}_{Q_1}\big| = \sum_{D \in \binom{I_1}{\ell}} |\mathcal{P}_{Q_1}(D)| \geq \sum_{D \in \binom{I_2}{k-\ell}} \left(|\mathcal{E}(D)| - \delta(|I_1|, \ell, Q_1)\binom{|I_1|}{\ell}\right) \geq \frac{\alpha}{4}|Z|, \tag{4}$$

The final inequality here holds since $\sum_{D\in\binom{I_2}{k-\ell}}|\mathcal{E}(D)| = |\mathcal{B}| \geq \frac{\alpha}{2}|Z|$ and $\alpha \geq 4\delta(|I_1|,\ell,Q_1)$. Lastly, using that $|\mathcal{P}_{Q_1}| \leq |I_1|^{2d_1}|Z|$ together with (4), we obtain $|(\mathcal{B}\times\mathcal{B})\cap\mathcal{P}_{Q_1}| \geq \frac{\alpha}{4|I_1|^{2d_1}}|\mathcal{P}_{Q_1}|$.

Now, from this bound we find a choice of $C, C' \in \binom{I_1}{\ell}$ with $\mathrm{pat}(C,C') = Q_1$ such that the set

$$\mathcal{F}_{C,C'} = \Big\{D \in \binom{I_2}{k-\ell} : C\cup D, C'\cup D \in \mathcal{B}\Big\}$$

satisfies $|\mathcal{F}_{C,C'}| \geq \frac{\alpha}{4|I_1|^{2d_1}}\binom{n_2}{k-\ell}$. However, if $D, D' \in \mathcal{F}$ with $\mathrm{pat}(D,D') = Q_2$ then $C\cup D, C'\cup D' \in \mathcal{A}$ and $\mathrm{pat}(C\cup D, C'\cup D') = Q_1Q_2 = P$. As $\mathcal{A}$ is P-free we see $\mathcal{F}_{C,C'} \subset \binom{I_2}{k-\ell}$ is Q_2-free. This gives $\frac{\alpha}{4|I_1|^{d_1}} \leq \delta(|I_2|, k-\ell, Q_2)$ and proves (3).

To complete the proof, note that as $\alpha \geq 2e^{-k_1/12}$, by definition of L we have $\ell \in L \subset [k_1, 2k_1]$ and $k-\ell \in [k_2, k_2+k_1]$. As $|I_1| = 3k_1$ and $|I_2| = 2k - 3k_1 = 2k_2 + k_1$, by Proposition 2.1 we find

$$\delta(|I_1|,\ell,Q_1) \leq \delta(2k_1,k_1,Q_1) = \delta(k_1,d_1) \qquad \text{and} \qquad \delta(|I_2|,k-\ell,Q_1) \leq \delta(2k_2,k_2,Q_2) = \delta(k_2,d_2).$$

Combined with (3) this completes the proof. □

Proof of Theorem 1. We prove by induction on d that with $a_d = (8d)^{5d}$ and $c_d = 6d8^{-d}$ we have

$$\delta(k,d) \leq a_d k^{-c_d}. \tag{5}$$

For $d=1$ we have $P = +-$ or $P = -+$ and $\mathcal{A} \subset \binom{[2k]}{k}$ is P-free simply means that $|A\triangle B| \neq 2$ for all distinct $A, B \in \mathcal{A}$. It is well known that such families satisfy $|\mathcal{A}| \leq \frac{1}{k}\binom{2k}{k}$. Indeed, for each $C \in \binom{[2k]}{k+1}$ let y_C denote the number of $A \in \mathcal{A}$ with $A \subset C$. Then

$$\sum_{C\in\binom{[2k]}{k+1}} y_C = \Big|\Big\{(A,C) \in \mathcal{A}\times\binom{[2k]}{k+1} : A\subset C\Big\}\Big| = |\mathcal{A}|\times k.$$

However, if $|A\triangle B| \neq 2$ for al distinctl $A, B \in \mathcal{A}$ we must have $y_C \leq 1$ for all C. Rearranging, we obtain the claimed upper bound on $|\mathcal{A}|$. This easily gives that (5) holds for $d=1$.

We now prove the result for a d-balanced pattern P, assuming by induction that the theorem holds for all d'-balanaced patterns with $d' < d$. We can assume that $k \geq a_d^{1/c_d} \geq 16^8$ as otherwise the statement is trivial. We will first prove this when P begins and ends with different signs, using Lemma 2.2, noting that in this range $k^{1/2} \geq 16\log k$. To apply this, let $\gamma = 8(a_{d-1})^{1/2}k^{-\frac{c_{d-1}}{4}}$ and note that $\gamma \geq 8(a_{d-1})^{1/2}k^{-\frac{1}{4}} \geq 16(\log k)k^{-1/2}$ since $k^{1/4}/\log k \geq 1/32 \geq 2(a_{d-1})^{-1/2}$. Therefore we can apply Lemma 2.2 to find

$$\begin{aligned}\delta(k,P) \leq \max\Big(\gamma, 6\sqrt{\delta(\lceil\tfrac{\gamma^2k}{64}\rceil, d-1)}\Big) &\leq \max\Big(8(a_{d-1})^{1/2}k^{-\frac{c_{d-1}}{4}}, 6\sqrt{a_{d-1}\big(a_{d-1}k^{1-\frac{c_{d-1}}{2}}\big)^{-c_{d-1}}}\Big)\\ &\leq 8(a_{d-1})^{1/2}k^{-\frac{c_{d-1}}{4}} \leq a_d k^{-c_d}.\end{aligned}$$

The second inequality here uses that Lemma 2.2 holds for $d-1$ by induction, the third that $(a_{d-1})^{-c_{d-1}} \leq 1$ and $1 - \frac{c_{d-1}}{2} \geq \frac{1}{2}$ and the last inequality uses that $c_d \leq \frac{c_{d-1}}{4}$.

We now move to the case where P starts and ends with the same signs. Given P let d_1 and d_2 be as in Lemma 2.3 so that $d_1 + d_2 = d$ with $d_i \geq 1$. We will assume that $d_1 \leq d_2$ as the other case follows similarly. Let us set $k_1 = \lceil k^\beta \rceil$ where $\beta = \frac{c_{d_2}}{2d_1 + c_{d_1}}$. Set $k_2 = k - 2k_1 \geq k - 4k^\beta \geq k - 4k^{1/2} \geq \frac{k}{2}$ for $k \geq 2^6$. Then by Lemma 2.3 we have

$$
\begin{aligned}
\delta(k, P) &\leq \max\left(2e^{-k_1/12}, 4\delta(k_1, d_1), 4(3k_1)^{2d_1}\delta(k_2, d_2)\right) \\
&\leq \max\left(2e^{-k^\beta/12}, 4a_{d_1} r^{-\beta c_{d_1}}, 4(6k^\beta)^{2d_1} a_{d_2}\left(\frac{k}{2}\right)^{-c_{d_2}}\right) \\
&\leq \max\left(2e^{-k^\beta/12}, 4a_{d_1} k^{-\beta c_{d_1}}, 8^{2d_1+3} a_{d_2} k^{2d_1\beta - c_{d_2}}\right) \\
&\leq \max\left(2e^{-k^\beta/12}, 8^{2d_1+3} a_{d_2} k^{-\frac{c_{d_1}c_{d_2}}{2d_1 + c_{d_1}}}\right) \leq a_d k^{-c_d}.
\end{aligned}
$$

The first part of the final inequality here uses $a_d \geq 2k^{c_d}$ for $k \leq (a_d/2)^{1/c_d}$ and that $e^{-k^\beta/12} \leq k^{-c_d}$ for $k \geq (a_d/2)^{1/c_d}$. The second part uses that $8^{2d_1+3}a_{d_2} \leq a_d$ and that since $d = d_1 + d_2$ and $d \leq 2d_2$ we have $c_d \leq 12d_2 8^{-d} \leq \frac{36 d_1 d_2 8^{-(d_1+d_2)}}{2d_1+1} \leq \frac{c_{d_1}c_{d_2}}{2d_1+1} \leq \frac{c_{d_1}c_{d_2}}{2d_1+c_{d_1}}$. This completes this case and the proof of the theorem. $\square$

3 Interval patterns

In this section, we first prove Theorem 2. We then give several lower bounds for the case $n = 2k$ depending on value of d.

3.1 Upper Bound on $\delta(n, n/2, \mathrm{IP}(d))$

Proof of Theorem 2. Let $m = \left\lfloor \frac{n}{8d^2} \right\rfloor$. We partition $[n]$ into m intervals, $[n] = I_1 \cup \cdots \cup I_m$ with $|I_i| = \lfloor 8d^2 \rfloor$ or $|I_i| = \lceil 8d^2 \rceil$ for all $i \in [m]$.

Consider the following way of choosing elements from $\binom{[n]}{n/2}$. First select a set $T \subset \binom{[n]}{n/2-d}$ uniformly at random. Let $J = \{i \in [m] : |I_i \setminus T| \geq d\}$. As $d < |I_i|/2$, for every $i \in [m]$ we have

$$
\mathbb{P}(i \in J) = \mathbb{P}(|I_i \setminus T| \geq d) > \mathbb{P}(|I_i \cap T| \leq |I_i|/2) \geq \frac{1}{2}.
$$

If $i \in J$, further select a set $S_i \subset \binom{I_i \setminus T}{d}$ uniformly at random, and set $A_i = T \cup S_i$. If $i \in [m] \setminus J$ simply set $A_i = \emptyset$.

Now for every $i, j \in J$ with $i < j$, we have $\mathrm{pat}(A_i, A_j) = \mathrm{IP}(d)$. Also for $i \notin J$ we have $A_i \notin \mathcal{A}$, since $|A_i| = 0 \neq n/2$. We conclude that there is at most one index $i \in [m]$ with $A_i \in \mathcal{A}$. Equivalently,

$$
\sum_{i=1}^m \mathbf{1}_{A_i \in \mathcal{A}} = \sum_{i=1}^m \sum_{A \in \mathcal{A}} \mathbf{1}_{A_i = A} \leq 1.
$$

This is true for any choice of T and S_i's, so in particular if we take the expectation on both sides, we have

$$
\sum_{i=1}^m \sum_{A \in \mathcal{A}} \mathbb{P}(A_i = A) \leq 1. \tag{6}
$$

But as $A_i \notin \mathcal{A}$ for $i \notin J$, given any $A \in \mathcal{A}$ we get that $\mathbb{P}(A_i = A) = \mathbb{P}(A_i = A | i \in J)\mathbb{P}(i \in J) > \frac{1}{2}\mathbb{P}(A_i = A | i \in J)$. Rewriting (6), this gives

$$\sum_{i=1}^{m} \sum_{A \in \mathcal{A}} \frac{\mathbb{P}(A_i = A | i \in J)}{2} \leq 1. \tag{7}$$

Lemma 3.1. *Let $A \in \binom{[n]}{n/2}$ be a fixed set. If $|A \cap I_i| \geq \frac{|I_i|}{2} + d$, then $\mathbb{P}(A_i = A | i \in J) \geq \frac{1}{\binom{n}{n/2}}$.*

Proof. Indeed, $\mathbb{P}(A_i = A | i \in J) = \frac{N_i(A)}{N_i}$ where

$$N_i(A) := \left| \left\{ (S_i, T) : S_i \in \binom{I_i}{d},\ T \in \binom{[n] \setminus S_i}{n/2 - d},\ S_i \cup T = A \right\} \right|;$$
$$N_i := \left| \left\{ (S_i, T) : S_i \in \binom{I_i}{d},\ T \in \binom{[n] \setminus S_i}{n/2 - d} \right\} \right|.$$

However, we have

$$\frac{N_i(A)}{N_i} \geq \frac{\binom{4d^2+d}{d}}{\binom{8d^2}{d}\binom{n-d}{\frac{n}{2}-d}} = \frac{(4d^2+d)_d(\frac{n}{2}-d)!\frac{n}{2}!}{(8d^2)_d(n-d)!} \geq \frac{(\frac{n}{2}-d)!\frac{n}{2}!}{2^d(n-d)!} > \frac{(n/2)!(n/2)!}{(n)!} = \frac{1}{\binom{n}{n/2}}.$$

$\square$

For a set $A \in \binom{[n]}{n/2}$, denote by $G(A) = \left|\left\{ i \in [m] : |A \cap I_i| \geq \frac{|I_i|}{2} + d \right\}\right|$. From Lemma 3.1 it follows that for any given A, we have $\sum_{i=1}^{m} \mathbb{P}\big(A_i = A | i \in J\big) \geq G(A) \times \frac{1}{\binom{n}{n/2}}$. Together with (7), we obtain

$$\sum_{A \in \mathcal{A}} G(A) \leq 2\binom{n}{n/2}. \tag{8}$$

We call a set $A \in \binom{[n]}{n/2}$ *bad*, if $G(A) < m/5$. Otherwise, we say that A is *good*. Let $\mathcal{B}$ be the family of all bad sets.

Lemma 3.2. $|\mathcal{B}| = o(\frac{1}{m}\binom{n}{n/2})$ *for sufficiently large n.*

Proof. For a uniform random choice of a set $A \subseteq \binom{[n]}{n/2}$, let X_i be a random variable, with $X_i = 1$ if $|A \cap I_i| > \frac{|I_i|}{2} + d$, and $X_i = 0$ otherwise. Let $Z = X_1 + \cdots + X_m$. To prove the lemma, we need to show that $\mathbb{P}(Z < m/5) = o(\frac{1}{m})$. By linearity of expectation, $\mathbb{E}Z = m\mathbb{E}X_i = m\mathbb{P}(X_i = 1)$. Notice that for every $i \neq j$, X_i and X_j are negatively correlated, since if A has many elements in one interval, it is less likely to have many elements on another interval.

$$\mathbb{P}(X_i = 0) = \frac{\sum_{i=0}^{4d^2+d}\binom{8d^2}{i}\binom{n-8d^2}{n/2-i}}{\binom{n}{n/2}} \leq \frac{1}{2} + \frac{\sum_{i=4d^2}^{4d^2+d}\binom{8d^2}{i}\binom{n-8d^2}{n/2-i}}{\binom{n}{n/2}} \leq \frac{1}{2} + \frac{d\binom{8d^2}{4d^2}\binom{n-8d^2}{n/2-4d^2}}{\binom{n}{n/2}} < 0.79.$$

The second inequality uses Stirling's formula. Therefore $\mathbb{P}(X_i = 1) = \mathbb{E}X_i > 0.21$. Using linearity of expectation gives $\mathbb{E}Z = \sum_{i=1}^{m} \mathbb{E}X_i > 0.21m$.

By a version of the Chernoff-Hoefding bound for negatively correlated variables [15], we deduce that $\mathbb{P}(A \in \mathcal{B}) = \mathbb{P}(Z < 0.2m) < \mathbb{P}(Z - \mathbb{E}Z > 0.01m) = o(\frac{1}{m})$, finishing the proof. $\square$

Therefore, if $|\mathcal{A}| \geq \frac{2}{m}\binom{n}{n/2}$, then $|\mathcal{A} \setminus \mathcal{B}| = (1-o(1))|\mathcal{A}|$. Using (8), we see that

$$(1-o(1))\frac{m|\mathcal{A}|}{10} \leq \sum_{A \in \mathcal{A}\setminus\mathcal{B}} G(A) \leq \sum_{A\in\mathcal{A}} G(A) \leq \binom{n}{n/2}. \tag{9}$$

Equivalently $|\mathcal{A}| = O\big(\frac{1}{m}\binom{n}{n/2}\big) = O\big(\frac{d^2}{n}\binom{n}{n/2}\big)$, as required. □

3.2 Lower Bound on $\delta(n, n/2, \mathrm{IP}(d))$

For the lower bounds, we provide different lower bounds, depending on the range of d.

Theorem 4. *The following hold:*

(i) If $d = o(\sqrt{n})$, there is an $\mathrm{IP}(d)$-free family $\mathcal{A} \subseteq \binom{[n]}{n/2}$ with $|\mathcal{A}| = \Omega(\max\{\frac{1}{nd}, \frac{d^2}{n^{3/2}}\} \cdot \binom{n}{n/2})$.

(ii) If $d = c\sqrt{n}$, there is an $\mathrm{IP}(d)$-free family $\mathcal{A} \subseteq \binom{[n]}{n/2}$ with $|\mathcal{A}| = \Omega_c(\binom{n}{n/2})$.

Proof. First we prove *(i)*. For a set $A \in \binom{[n]}{n/2}$ let $S(A) := \sum_{i\in A} i$, the sum of the elements in A. Observe that if $\mathrm{pat}(A,B) = \mathrm{IP}(d)$ then $0 < |S(A) - S(B)| < nd$. Thus for any $0 \leq i \leq nd-1$, the family $\mathcal{A}_i := \big\{A \in \binom{[n]}{n/2} |\, S(A) \equiv i (\mod nd)\big\}$ forms an $\mathrm{IP}(d)$-free family. By the pigeonhole principle, we can find such i so that $|\mathcal{A}_i| \geq \frac{1}{nd}\binom{n}{n/2}$.

To obtain the second bound from *(i)*, note that if we choose a set $A \in \binom{[n]}{n/2}$ uniformly at random,

$$\mathbb{E}[S(A)] = \frac{n(n+1)}{4}. \tag{10}$$

To calculate the variance, let

$$X_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } n \notin A. \end{cases}$$

Then $S(A) = \sum_{i=1}^n iX_i$. Now $\mathbb{E}[X_i] = \frac{1}{2}$ every $i \in [n]$ and $\mathbb{E}[X_iX_j] = \frac{1}{4}(1 - \frac{1}{n-1})$. Using this, we find

$$\begin{aligned}\mathbb{V}\mathrm{ar}(S(A)) = \mathbb{E}[(\sum_{i=1}^n iX_i)^2] - \mathbb{E}[\sum_{i=1}^n iX_i)]^2 &\leq \sum_{i\in[n]} i^2\mathbb{E}[X_i] + \sum_{i\neq j} ij\big(\mathbb{E}[X_iX_j] - \mathbb{E}[X_i]\mathbb{E}[X_j]\big) \\ &\leq \sum_{i\in[n]} \frac{i^2}{2} \leq \frac{n^3}{2}. \end{aligned} \tag{11}$$

From (10) and (11) together, by Chebyshev's inequality we get $\mathbb{P}(|S(A) - n(n+1)/4| \leq n^{3/2}) \geq 1/2$. Equivalently, $|\{A \in \binom{[n]}{n/2} : |S(A) - n(n+1)/4| \leq n^{3/2}\}| \geq \frac{1}{2}\binom{n}{n/2}$. By an easy averaging argument, for some value $m \in [\frac{n(n+1)}{4} - \frac{n^{3/2}}{3}, \frac{n(n+1)}{4} + \frac{n^{3/2}}{2}]$.

$$|\{A \in \binom{[n]}{n/2} : S(A) \in [m - \frac{d^2}{2}, m + \frac{d^2}{2})\}| \geq \frac{1}{2(2n^{3/2}/d^2 + 2)}\binom{n}{n/2} = \Omega\Big(\frac{d^2}{n^{3/2}}\binom{n}{n/2}\Big)$$

However, since two sets $A, B \in \binom{n}{n/2}$ with $\mathrm{pat}(A,B) = \mathrm{IP}(d)$ have $|S(A) - S(B)| > d^2$, this completes the proof of *(i)*.

To prove *(ii)*, let $c > 0$ be given and let $d = c\sqrt{n}$. Note that if $pat(A, B) = IP(d)$ then for some $i \in [n]$ we have $|A \cap [i]| \geq |B \cap [i]| + d$. This shows that $\mathcal{A} = \{A \in \binom{[n]}{n/2} : \big||A \cap [i]| - i/2\big| < d/4 \text{ for all } i \in [n]\}$ is an $IP(d)$-free family. We will now show that $|\mathcal{A}| = \Omega_c(\binom{n}{n/2})$.

To see this, it is convenient to identify elements of $\binom{[n]}{n/2}$ with certain walks. Let $\mathcal{W}_0$ denote the set of all walks $W = W_0 \cdots W_n$ of length n on $\mathbb{Z}$ with $W_0 = W_n = 0$ and which either increase or decrease by 1 in each step (i.e. $|W_i - W_{i-1}| = 1$ for all $i \in [n]$). Note that each walk $W \in \mathcal{W}_0$ naturally corresponds to a subset of $[n]$ of size $n/2$ consisting of those steps in $[n]$ where the walk increases. Under this correspondence, the set $\mathcal{A}$ corresponds to those walks in $\mathcal{W}_0$ which lie entirely in $[-d/4, d/4]$.

Now select a walk $W \in \mathcal{W}_0$ uniformly at random. Letting T denote a value to be determined, consider the following events:

$$\begin{aligned} A &= \{W_j \in [-d/4, d/4] \text{ for all } j \in [n]\} \\ B &= \{W_{in/T} \in [-d/12, d/12] \text{ for all } i \in [T-1]\} \\ C_i &= \{W_j \in [-d/4, d/4] \text{ for all } j \in [\frac{(i-1)n}{T}, \frac{in}{T}]\}, \text{ where } i \in [T]. \end{aligned}$$

Also for $i \in [T-1]$ and $a_i \in [-d/12, d/12]$, let $B_i(a_i)$ denote the event $B_i(a_i) = \{W_{in/T} = a_i\}$. We will show that

$$\mathbb{P}_{W \sim \mathcal{W}_0}\Big(B \wedge \bigwedge_{i \in [T]} C_i\Big) \geq c' > 0, \tag{12}$$

where c' depends only on c. Since $\bigwedge_{i \in [T]} C_i \subset A$, this will prove the result.

To begin, note that we have

$$\begin{aligned} \mathbb{P}_{W \sim \mathcal{W}_0}\Big(B \wedge \bigwedge_{i \in [T]} C_i\Big) &\geq \sum_{a_1, \ldots, a_{T-1} \in [-d/12, d/12]} \mathbb{P}_{W \sim \mathcal{W}_0}\Big(\bigwedge_{i \in [T-1]} B_i(a_i) \wedge \bigwedge_{i \in [T]} C_i\Big) \\ &= \sum_{a_1, \ldots, a_{T-1} \in [-d/12, d/12]} \mathbb{P}_{W \sim \mathcal{W}_0}\Big(\bigwedge_{i \in [T]} C_i \big| \bigwedge_{i \in [T-1]} B_i(a_i)\Big) \\ &\qquad \times \mathbb{P}_{W \sim \mathcal{W}_0}\Big(\bigwedge_{i \in [T-1]} B_i(a_i)\Big). \end{aligned} \tag{13}$$

Let $\mathcal{W}(a, b)$ denote the collection of random walks of length n/T which start at a and end at b. Since C_i depends only on $\{W_j : j \in [(i-1)n/T, in/T]\}$, taking $a_0 = a_T = 0$ we have

$$\begin{aligned} \mathbb{P}_{W \sim \mathcal{W}_0}\Big(\bigwedge_{i \in [T]} C_i \big| \bigwedge_{i \in [T-1]} B_i(a_i)\Big) &= \prod_{i \in [T]} \mathbb{P}_{W \sim \mathcal{W}_0}\Big(C_i \big| B_{i-1}(a_{i-1}) \wedge B_i(a_i)\Big) \\ &= \prod_{i \in [T]} \mathbb{P}_{W \sim \mathcal{W}(a_{i-1}, a_i)}\Big(W \text{ lies entirely in } [-d/4, d/4]\Big). \end{aligned} \tag{14}$$

Claim: For every $a, b \in [-d/12, d/12]$ we have $\mathbb{P}_{W \sim \mathcal{W}(a,b)}\Big(W \text{ lies entirely in } [-d/4, d/4]\Big) \geq 1/2$.

Let $\mathcal{W}(a)$ denote the collection of all walks of length n/T which begin at a. Let us select W from $\mathcal{W}(a)$ uniformly at random and let $S_{n/T}$ denote the final vertex. By the reflection principle for random walks, we have

$$\begin{aligned}
\mathbb{P}_{W\sim\mathcal{W}(a,b)}(W \text{ exceeds } d/4) &= \mathbb{P}_{W\sim\mathcal{W}(a)}(W \text{ exceeds } d/4 | S_{n/T} = b)\\
&= \frac{\mathbb{P}_{W\sim\mathcal{W}(a)}(S_{n/T} = d/2-b)}{\mathbb{P}_{W\sim\mathcal{W}(a)}(S_{n/T} = b)}\\
&= \frac{\binom{n/T}{n/2T+(d/2-b)-a}}{\binom{n/T}{n/2T+b-a}} \leq \frac{\binom{n/T}{n/2T+d/3}}{\binom{n/T}{n/2T+d/6}}\\
&= \frac{(n/2T-d/6)_{d/6}}{(n/2T+d/6)_{d/6}} \leq \Big(1-\frac{dT}{3n}\Big)^{d/6} \leq e^{-d^2T/36n}.
\end{aligned}$$

Taking $T = 72/c^2$ say, we find $\mathbb{P}_{W\sim\mathcal{W}(a,b)}(W \text{ exceeds } d/4) \leq e^{-2} < 1/4$. By symmetry, this gives $\mathbb{P}_{W\sim\mathcal{W}(a,b)}\Big(W \text{ lies entirely in } [-d/4, d/4]\Big) \geq 1 - 2\times(1/4) = 1/2$, as claimed.

Now by combining (14) together with the claim in (13) we find

$$\mathbb{P}_{W\sim\mathcal{W}_0}\Big(B \wedge \bigwedge_{i\in[T]} C_i\Big) \geq \sum_{a_1,\ldots,a_{T-1}\in[-d/12,d/12]} 2^{1-T}\times \mathbb{P}_{W\sim\mathcal{W}_0}\Big(\bigwedge_{i\in[T-1]} B_i(a_i)\Big). \tag{15}$$

But letting $b_i := \frac{n}{2T} + a_i - a_{i-1}$ for all $i \in [T]$ where $a_0 = a_T = 0$, we have

$$\mathbb{P}_{W\sim\mathcal{W}_0}\Big(\bigwedge_{i\in[T-1]} B_i(a_i)\Big) = \frac{\prod_{i\in[T]}\binom{n/T}{b_i}}{\binom{n}{n/2}} = \Omega_{c,T}(d^{1-T}).$$

The final inequality follows by Stirling's approximation, using that $b_i \in [\frac{n}{2T} - \frac{d}{6}, \frac{n}{2T} + \frac{d}{6}]$ for all $i \in [T]$. Combined with (15), this gives $\mathbb{P}_{W\sim\mathcal{W}_0}\Big(B \wedge \bigwedge_{i\in[T]} C_i\Big) = \Omega_{c,T}(1) = \Omega_c(1)$, as required. □

4 Alternating patterns

To begin, we prove an auxiliary lemma. Given $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i)$ in $[m]^D$ we say that $\mathbf{y}$ d-dominates $\mathbf{x}$ if $|\{i \in [D] : x_i \neq y_i\}| = d$ and $x_i \leq y_i$ for all $i \in [D]$.

Lemma 4.1. *Let $d, m, D \in \mathbb{N}$ with $2md^2 \leq D$. Suppose that $\mathcal{C} \subset [m]^D$ does not contain $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{y}$ d-dominates $\mathbf{x}$. Then $|\mathcal{C}| \leq 2m^{D-1}$.*

Proof. To begin, choose a set $S \subset [D]$ with $|S| = d$ and a vector $\mathbf{z} \in [m]^{[D]\setminus S}$ uniformly at random. For each $i \in [m]$ let $\mathbf{z}_S(i) \in [m]^D$ denote the vector which agrees with $\mathbf{z}$ on coordinates in $[D]\setminus S$ and equals i everywhere else. Also let $\mathcal{B}_{S,\mathbf{z}}$ denote the combinatorial line $\mathcal{B}_{S,\mathbf{z}} := \{\mathbf{z}_S(i) : i \in [m]\}$.

Now as $\mathcal{C}$ does not contain any d-dominating pairs, for any choice of S and $\mathbf{z}$ we have $|\mathcal{C} \cap \mathcal{B}_{S,\mathbf{z}}| \leq 1$. Letting X_i denote the indicator random variable which is 1 if $\mathbf{z}_S(i) \in \mathcal{C}$ and 0 otherwise, this gives

$$\sum_{i\in[m]} X_i \leq 1.$$

Taking expectations over all choice of S and $\mathbf{z}$, this gives

$$\sum_{C\in\mathcal{C}} \mathbb{P}(C \in \mathcal{B}_{S,\mathbf{z}}) = \sum_{i\in[m]}\sum_{C\in\mathcal{C}} \mathbb{P}(\mathbf{z}_S(i) = C) \leq 1. \tag{16}$$

However, an easy calculation gives that if C has k_i entries i for all $i \in [m]$, then

$$\mathbb{P}(C \in \mathcal{B}_{S,\mathbf{z}}) = \sum_{i\in[m]} \frac{\binom{k_i}{d}}{m^{D-d}\binom{D}{d}}.$$

This expression is minimized when all k_i are as equal as possible. Thus

$$\begin{aligned}\mathbb{P}(C \in \mathcal{B}_{S,\mathbf{z}}) \geq m\frac{\binom{D/m}{d}}{m^{D-d}\binom{D}{d}} = m\frac{(D/m)_d}{m^{D-d}D_d} &= \frac{m}{m^D}\prod_{l\in[0,d-1]}\Big(1-\frac{l(m-1)}{D-l}\Big)\\ &\geq \frac{1}{m^{D-1}}\Big(1-\sum_{l\in[0,d-1]}\frac{l(m-1)}{D/2}\Big)\\ &\geq \frac{1}{m^{D-1}}\Big(1-\frac{md^2}{D}\Big) \geq \frac{1}{2m^{D-1}}.\end{aligned}$$

The final line here used $2md^2 \leq D$. Combined with (16) this gives $|\mathcal{C}|/2m^{D-1} \leq 1$, as required. $\square$

We are now ready for the proof of Theorem 3.

Proof of Theorem 3. By Proposition 2.1 it suffices to prove the theorem for $n = 2k$. Let $m = \lfloor\frac{\log_2(n/d^2)}{2}\rfloor$. For convenience we assume that n is divisible by m, with $Km = n$. Let $[n] = \bigcup_{i=1}^K I_i$ be a partition of $[n]$ where $I_i = \{(i-1)m+1,\ldots,im\}$ for all $i \in [K]$. Given a set $T \subset [K]$, let $T^c = [K]\setminus T$ and let

$$\mathcal{B}_T := \{A \subset \bigcup_{i\in T^c} I_i : |A\cap I_i| \neq 1 \text{ for all } i \in T^c\}.$$

Given $B \in \mathcal{B}_T$ and $\mathbf{x} \in [m]^T$ we also let $B(\mathbf{x}) := B \cup \{(i-1)m+j-1 : i \in T, x_i = j\}$ and

$$\mathcal{C}_B := \{B(\mathbf{x}) : \mathbf{x} \in [m]^T\}.$$

Note that for every $A \subset [n]$ there is a unique $T \subset [K]$, $B \in \mathcal{B}_T$ and $\mathbf{x} \in [m]^T$ such that $A = B(\mathbf{x})$. Thus we have the disjoint union

$$\binom{[n]}{n/2} = \bigcup_{T\subset[K]} \bigcup_{\substack{B\subset\mathcal{B}_T\\ |B|=\frac{n}{2}-|T|}} \mathcal{C}_B. \tag{17}$$

We will first show that almost all sets A in $\binom{[n]}{n/2}$ are of the form $A = B(\mathbf{x})$ where $T \subset [K]$ and $B \in \mathcal{B}_T$ with $|T| \geq mK/2^{m+1} = n/2^{m+1}$. To see this, given a set $A \subset [n]$, let $A_i = A \cap I_i$ for all $i \in [K]$. We will say that $A \subset [n]$ is *bad* if $T(A) = \{i \in [K] : |A_i| = 1\}$ satisfies $|T(A)| \leq \frac{m}{2^{m+1}}K$. We claim that there are at most $O(e^{-n^{1/2}/2}2^n)$ sets are bad. Indeed, if we select $A \subset [n]$ uniformly at random, we have $\mathbb{P}(|A_i| = 1) = m/2^m$, which gives $\mathbb{E}(|T(A)|) = \frac{mK}{2^m} = \frac{n}{2^m}$. As $|A_i| = 1$ for each $i \in [K]$ independently, by Chernoff's inequality, we find that $\mathbb{P}\big(|T(A)| - \frac{n}{2^m} \leq -\frac{n}{2^{m+1}}\big) \leq e^{-\frac{n}{2^{m+1}}}$. As $m \leq \log_2(n/d^2)/2 \leq \frac{\log_2 n}{2}$ we find that $\mathbb{P}(A \text{ is bad}) \leq e^{-n^{1/2}/2}$. Equivalently, $|\{A \subset [n] : A \text{ is bad}\}| = O(e^{-n^{1/2}}2^n)$.

Now suppose that $T \subset [K]$ with $|T| \geq n/2^{m+1}$ and $B \in \mathcal{B}_T$. Note that given $\mathbf{x}, \mathbf{y} \in [m]^T$, if $\mathbf{y}$ d-dominates $\mathbf{x}$ then $\mathrm{pat}(B(\mathbf{x}), B(\mathbf{y})) = \mathrm{AP}(d)$. Noting that as $m = \lfloor \log_2(n/d^2)/2 \rfloor$ we have $|T| \geq n/2^{m+1} \geq 2^m d^2 \geq 2md^2$. Setting $D = |T|$, Lemma 4.1 therefore shows that any $\mathcal{A} \subset \binom{[n]}{n/2}$ which is $\mathrm{AP}(d)$-free satisfies

$$|\mathcal{A} \cap \mathcal{C}_B| \leq 2m^{|T|-1} = \frac{2}{m}|\mathcal{C}_B|. \tag{18}$$

Summing over all $T \subset [K]$ and $B \in \mathcal{C}_T$, combined with (17) and (18), this gives

$$\begin{aligned} |\mathcal{A}| \leq \sum_{T \subset [K]} |\mathcal{A} \cap \bigcup_{\substack{B \in \mathcal{B}_T \\ |B| = n/2 - |T|}} \mathcal{C}_B| &\leq \big|\{A \subset [n] : A \text{ bad}\}\big| + \sum_{\substack{T \subset [K]: \\ |T| \geq 2md^2}} \sum_{B \in \mathcal{B}_T} |\mathcal{A} \cap \mathcal{C}_B| \\ &\leq O\Big(\frac{2^n}{e^{\sqrt{n}/2}}\Big) + \sum_{\substack{T \subset [K]: \\ |T| \geq 2md^2}} \sum_{\substack{B \in \mathcal{B}_T \\ |B| = n/2 - |T|}} \frac{2}{m}|\mathcal{C}_B| \\ &\leq \frac{2 + o(1)}{m}\binom{n}{n/2}. \end{aligned}$$

This completes the proof of the theorem. □

5 Concluding remarks and open problems

In this paper we proved bounds on the size of families $\mathcal{A} \subset \mathcal{P}[n]$ which avoid a d-balanced pattern P. Our proof shows that such families satisfy

$$|\mathcal{A}| = O(a_d n^{-c_d} 2^n),$$

where $a_k = (8d)^{5d}$ and $c_d = 6d8^{-d}$. In particular, families $\mathcal{A}$ which avoid a d-balanced pattern with $d < c \log\log n$ satisfy $|\mathcal{A}| = o(2^n)$ for some absolute constant $c > 0$. It would be interesting to improve the density bound here and/or extend the range of d for which this zero density property holds.

Another interesting question is the following: which balanced pattern P has the strongest effect on the density of P-free families $\mathcal{A} \subset \mathcal{P}[n]$? That is, what is $\min_P \delta(n,k,P)$, where the minimum is taken over all balanced patterns P? If instead of patterns we only forbid intersection sizes (as discussed in the Introduction) then there are a number of very strong density results for subsets of $\mathcal{P}[n]$. For example, the Frankl-Rödl [10] theorem shows that given $\epsilon > 0$, if $\mathcal{A} \subset \mathcal{P}[n]$ and $|A \cap B| \neq t$ for some $\epsilon n \leq t \leq (1/2 - \epsilon)n$ then $|\mathcal{A}| \leq (2-\delta)^n$, where $\epsilon = \epsilon(\delta) > 0$. It would be very interesting to know if there exists a pattern which forces a superpolynomial density in n. That is, does there an increasing sequence of naturals $(n_k)_{k \in \mathbb{N}}$ and balanced patterns $(P_k)_k$ with $\delta(n_k, n_k/2, P_k) = n_k^{-\omega_k(1)}$ for some function $\omega_k(1)$ tending to infinity with k?

Lastly, how large can d be (as a function of n) while still giving $\delta(n, n/2, \mathrm{AP}(d)) \to 0$ as $n \to \infty$. Theorem 3 proves that this holds for any $d = o(\sqrt{n})$.

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